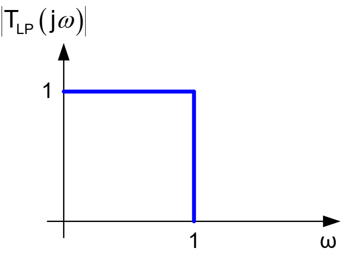
EE 508
Lecture 10

The Approximation Problem

Classical Approximations
– the Chebyschev Approximations

Review from Last Time

Butterworth Approximations



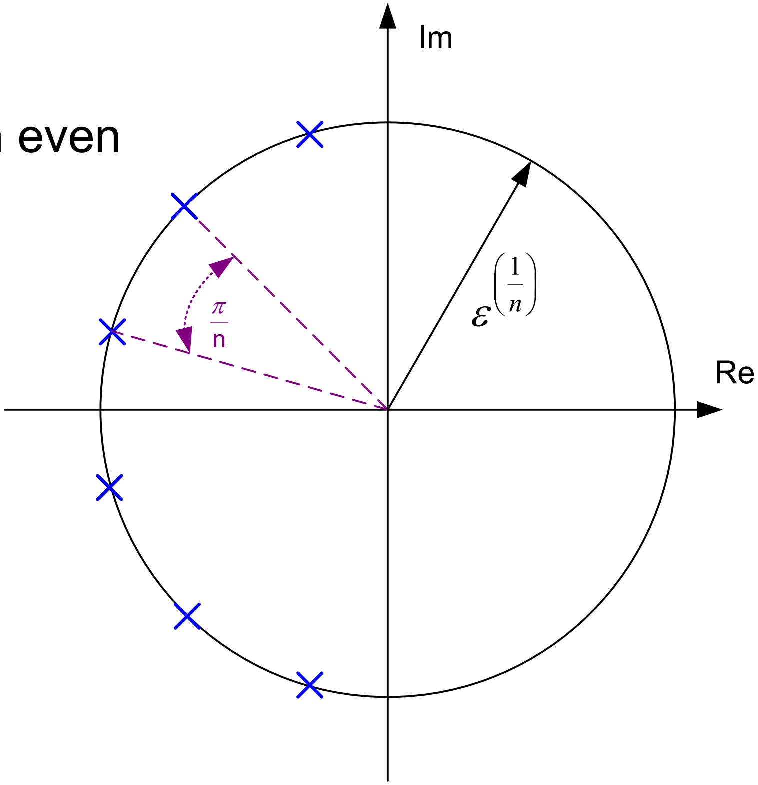
- Analytical formulation:
  - All pole approximation
  - Magnitude response is maximally flat at ω=0
  - Goes to 0 at ω=∞
  - Assumes value $\sqrt{\frac{1}{1+\varepsilon^2}}$ at ω=1
  - Assumes value of 1 at ω=0
  - Characterized by {n,ε}
- Emphasis almost entirely on performance at single frequency

"On the Theory of Filter Amplifiers", Wireless Engineer (also called Experimental Wireless and the Radio Engineer), Vol. 7, 1930, pp. 536-541.

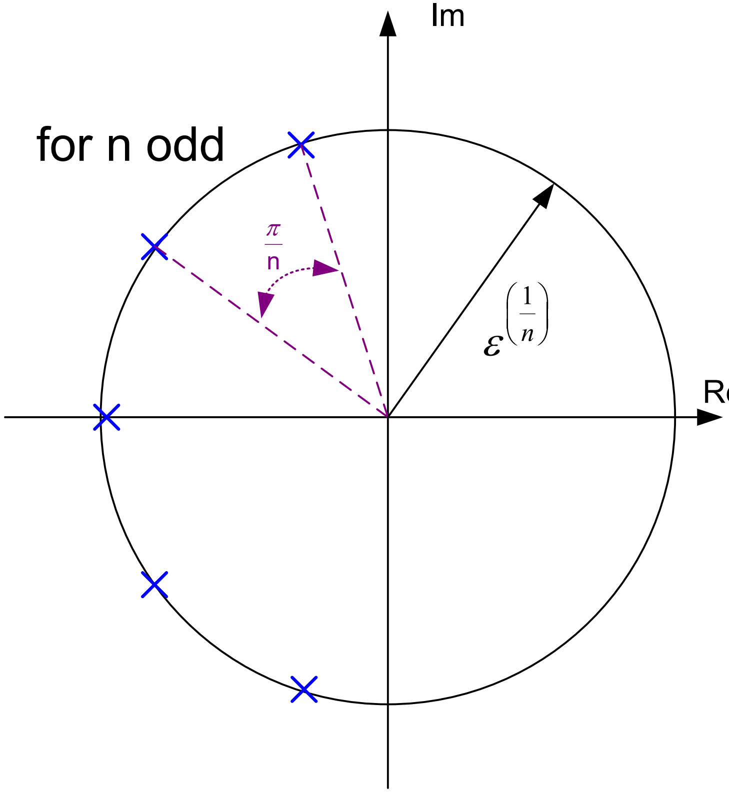
Butterworth Approximation

Poles of $T_{BW}(s)$

for n even



for n odd



$$p_{k+1} = \varepsilon^{1/n}\left[-\sin\left(\left[1+2k\right]\frac{\pi}{2n}\right) \pm j\cos\left(\left[1+2k\right]\frac{\pi}{2n}\right)\right]$$

$$k=0,1,\ \ldots\ \frac{n}{2}-1$$

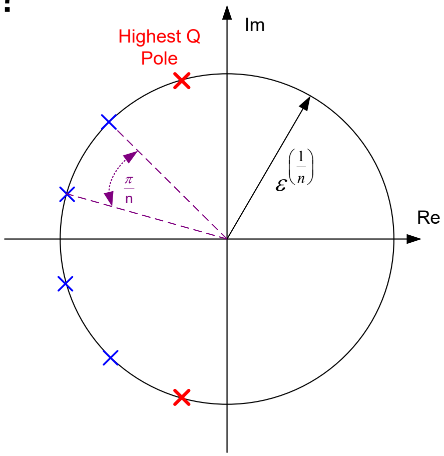
$$p_n = \varepsilon^{1/n}\left[-1+j0\right]$$

$$p_k = \varepsilon^{1/n}\left[-\sin\left(\left[1+2k\right]\frac{\pi}{2n}\right) \pm j\cos\left(\left[1+2k\right]\frac{\pi}{2n}\right)\right] \quad k=0,\ \ldots\ \frac{n-3}{2}$$

Butterworth Approximation

What is the Q of the highest Q pole for the BW approximation?

$$p_0 = \varepsilon^{1/n}\left[-\sin\left(\frac{\pi}{2n}\right) + j\cos\left(\frac{\pi}{2n}\right)\right] = \alpha + j\beta$$



$$Q_{MAX} = \frac{\sqrt{\alpha^2 + \beta^2}}{-2\alpha}$$

$$Q_{MAX} = \frac{\varepsilon^{1/n}\sqrt{\sin^2\left(\frac{\pi}{2n}\right) + \cos^2\left(\frac{\pi}{2n}\right)}}{2\varepsilon^{1/n}\sin\left(\frac{\pi}{2n}\right)} = \frac{1}{2\sin\left(\frac{\pi}{2n}\right)}$$

$$Q_{MAX} = \frac{1}{2\sin\left(\frac{\pi}{2n}\right)}$$

Butterworth Approximation

Review from Last Time

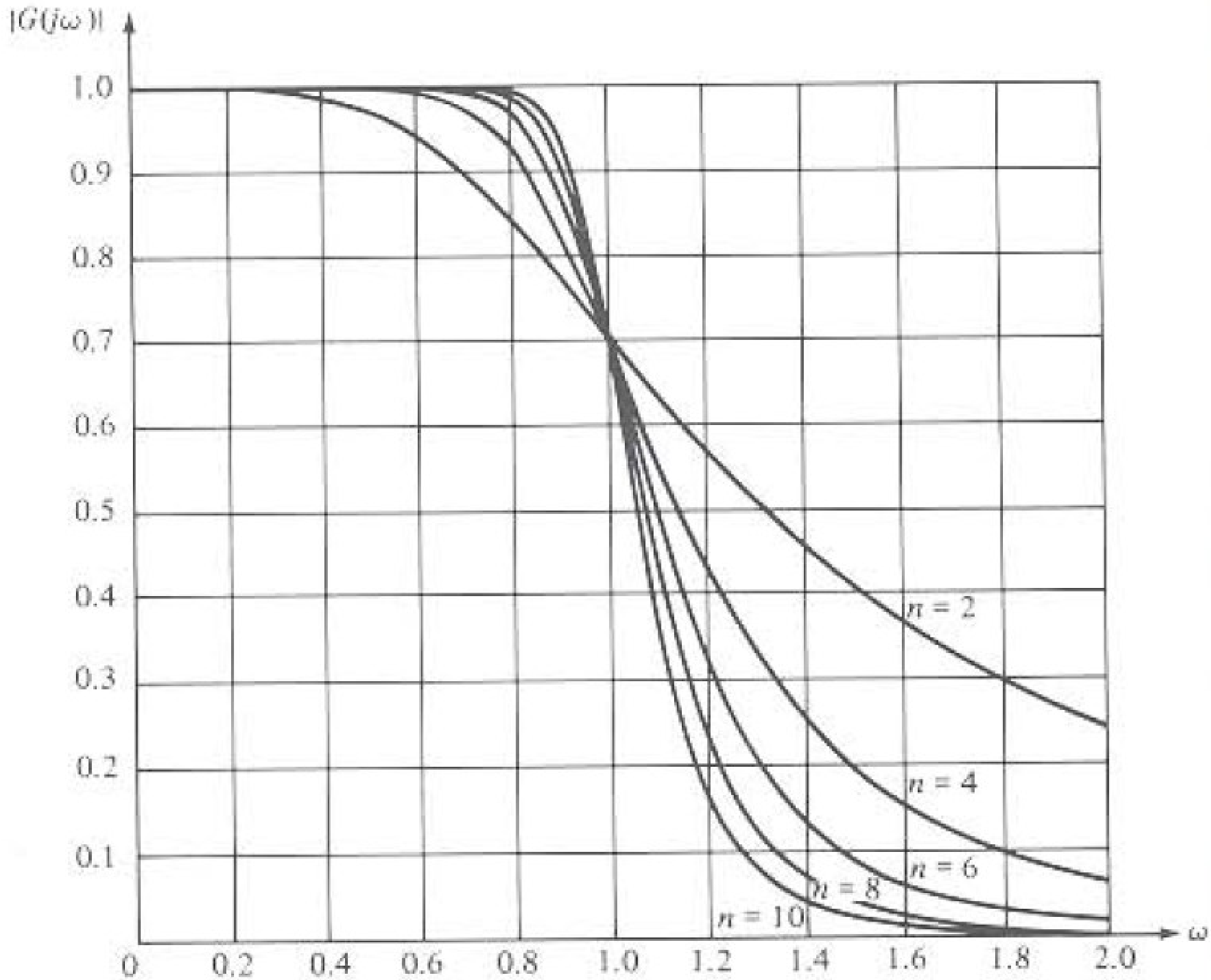


Fig. 17-3a Magnitude of the maximally flat approximation ($\varepsilon = 1$)

Figure from Passive and Activ
Network Analysis and
Synthesis, Budak

Order needs to be rather high to get steep transition

Butterworth Approximation

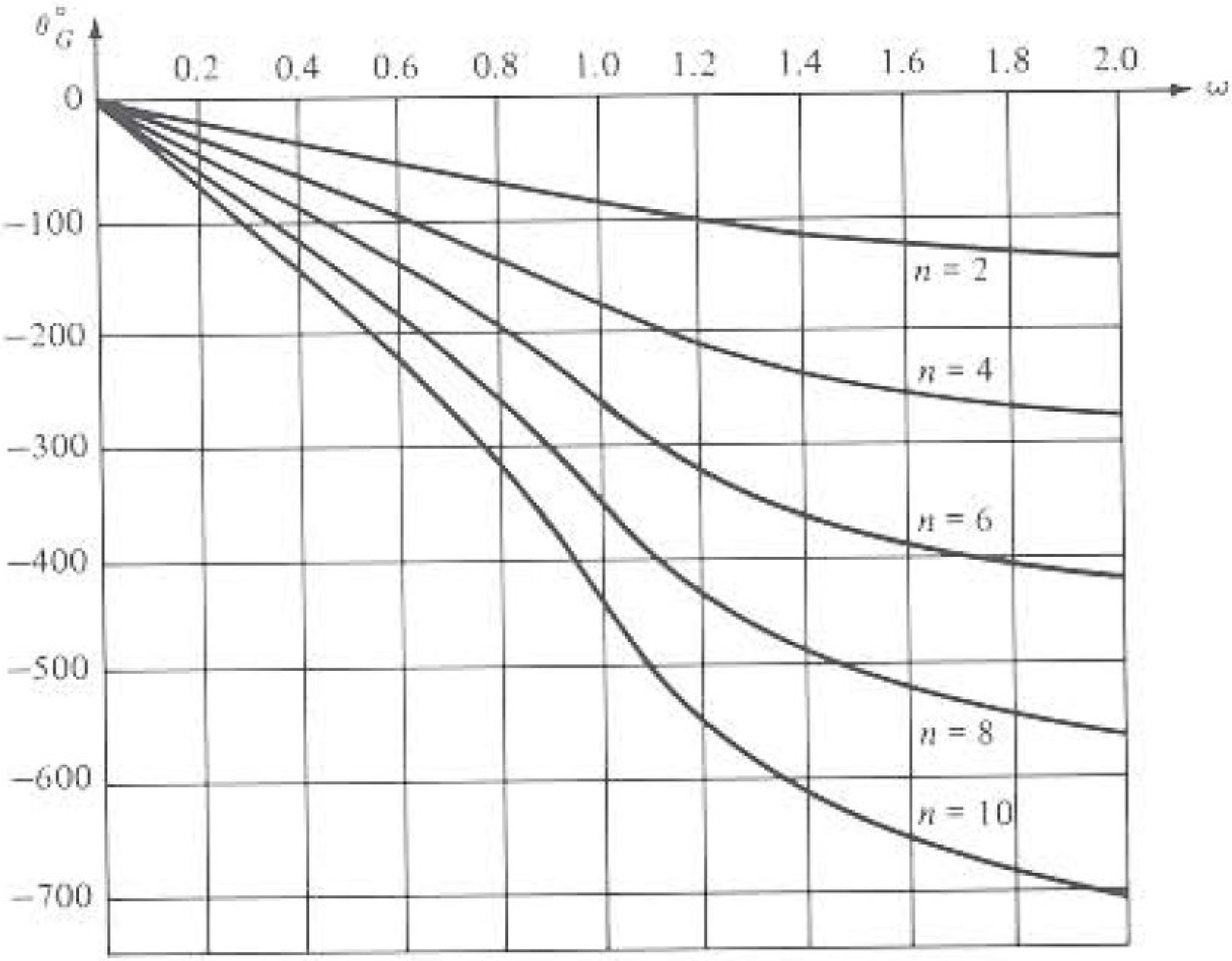


Fig. 17-3b Phase of the maximally flat approximation ($\varepsilon = 1$)

Figure from Passive and Active Network Analysis and Synthesis, Budak

Phase is quite linear in passband (benefit unrelated to design requirements)

Butterworth Approximation

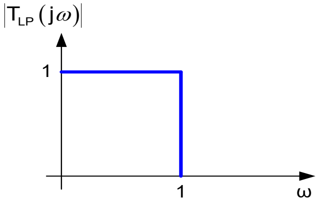
Review from Last Time

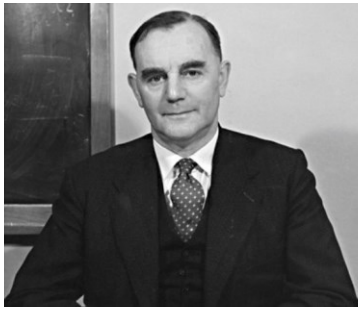
Summary

- Widely Used Analytical Approximation
- Characterized by {ε,n}
- Maximally flat at ω=0
- Almost all emphasis placed on characteristics at single frequency (ω=0)
- Transition not very steep (requires large order for steep transition)
- Pole Q is quite low
- Pass-band phase is quite linear (no emphasis was placed on phase!)
- Poles lie on a circle
- Simple closed-form analytical expressions for poles and |T(jω)|

Approximations

- Magnitude Squared Approximating Functions – $H_A(\omega^2)$
- Inverse Transform - $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares Approximations
- Pade Approximations
- Other Analytical Optimizations
- Numerical Optimization
- Canonical Approximations
  - Butterworth
  - Chebyschev
  - Elliptic
  - Bessel
  - Thomson

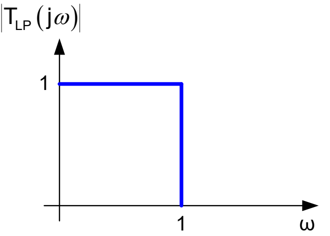




Stephen Butterworth
1885-1958

Pafnuty Lvovich Chebyshev
Born May 16, 1821
Died December 8,1894
Nationality Russian
Fields Mathematician

Chebyshev Approximations



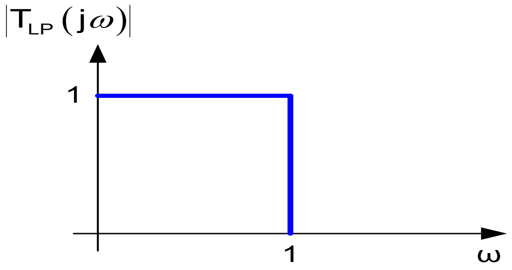
Type I Chebyshev Approximations

- Analytical formulation:
  - All pole approximation of order n
  - Magnitude response bounded between 1 and $\sqrt{\frac{1}{1+\varepsilon^2}}$ in the pass band
  - Assumes the value of $\sqrt{\frac{1}{1+\varepsilon^2}}$ at ω=1
  - Goes to 0 at ω=∞
  - Assumes extreme values maximum no times in [0 1]
  - Characterized by {n,ε}
- Based upon Chebyshev Polynomials

Chebyshev polynomials were first presented in: P. L. Chebyshev (1854) "Théorie des mécanismes connus sous le nom parallelogrammes," *Mémoires des Savants étrangers présentes à l'Academie de Saint-Pétersbourg*, vol. 7, pages 539-586.

Chebyschev had nothing to do with the design of filters but others applied his mathematical results to the filters field!

Chebyshev Approximations



Type II Chebyshev Approximations (not so common)

- Analytical formulation:
  - Magnitude response bounded between 0 and $\frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ in the stop band
  - Assumes the value of $\sqrt{\frac{1}{1+\varepsilon^2}}$ at ω=1
  - Order = n
  - Value of 1 at ω=0
  - Assumes extreme values maximum times in [1 ∞]
  - Characterized by {n,ε}
- Based upon Chebyshev Polynomials

Chebyshev Approximations

Chebyshev Polynomials

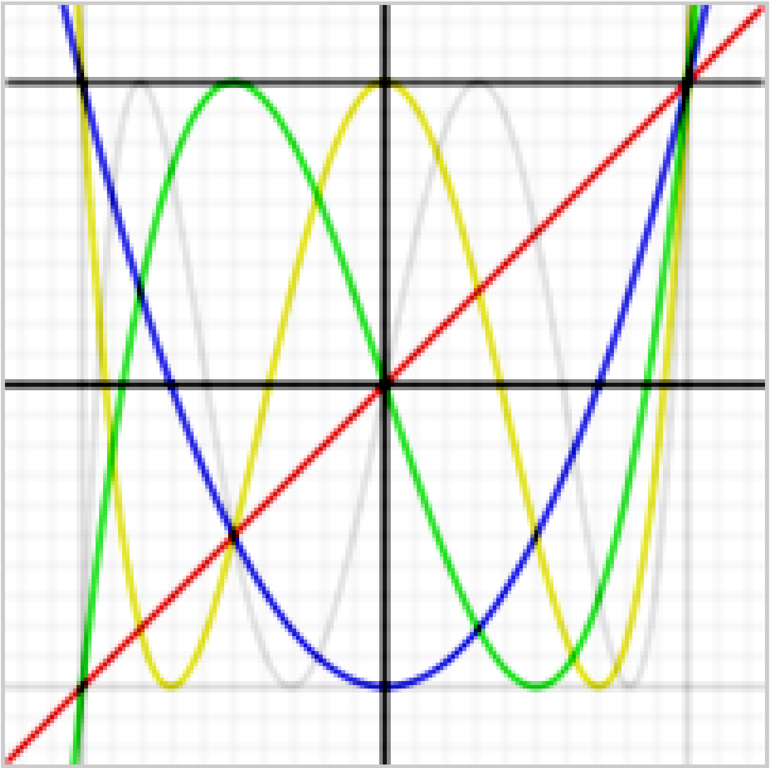
The Chebyshev polynomials are characterized by the property that the magnitude of the polynomial assumes the extremum values of 0 and 1 a maximum number of times in the interval [-1,1] and goes to $\pm\infty$ for |x| large.

In polynomial form they can be expressed as

$$C_0(x)=1$$
$$C_1(x)=x$$
$$C_{n+1}(x)=2xC_n(x) - C_{n-1}(x)$$

Or, equivalently, in trigonometric form as

$$C_n(x)=\begin{cases}\cos(n \bullet arc\cos(x)) & x\in[-1,1]\\ \cosh(n \bullet arc\cos h(x)) & x\geq 1\\ (-1)^n \cosh(n \bullet arc\cos h(-x)) & x\leq -1\end{cases}$$



This image shows the first few Chebyshev polynomials of the first kind in the domain $-1¼ < x < 1¼$, $-1¼ < y < 1¼$; the flat T_0, and T_1, T_2, T_3, T_4 and T_5.

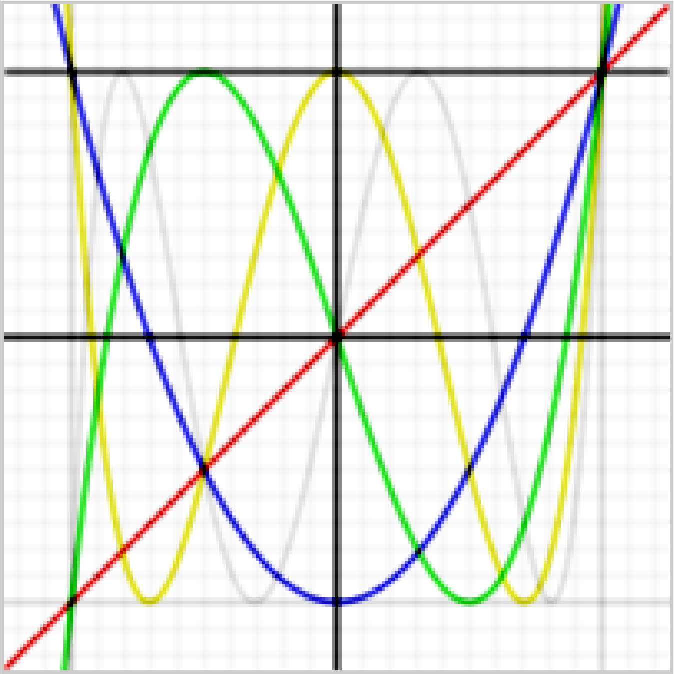
Figure from Wikipedia

Chebyshev Approximations

Chebyshev Polynomials

The first 9 CC polynomials:

$$C_0(x) = 1$$
$$C_1(x) = x$$
$$C_2(x) = 2x^2 - 1$$
$$C_3(x) = 4x^3 - 3x$$
$$C_4(x) = 8x^4 - 8x^2 + 1$$
$$C_5(x) = 16x^5 - 20x^3 + 5x$$
$$C_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$
$$C_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$
$$C_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$



This image shows the first few Chebyshev polynomials of the first kind in the domain $-1¼ < x < 1¼$, $-1¼ < y < 1¼$; the flat T_0, and T_1, T_2, T_3, T_4 and T_5. Figure from Wikipedia

- Even-indexed polynomials are functions of x^2
- Odd-indexed polynomials are product of x and function of x^2
- Square of all polynomials are function of x^2 (i.e. an even function of x)

Chebyshev Approximations

Type 1

$$H_{BW}(\omega)=\frac{1}{1+\varepsilon^2\omega^{2n}}$$

Butterworth

$$H(\omega)=\frac{1}{1+\varepsilon^2 F_n\left(\omega^2\right)}$$

A General Form for Low-pass Filter

Desired Characteristics of General Form of LP filters (derived from BW observations):

- $F_n(\omega^2)$ close to 1 in the pass band and gets very large in stop-band
- These characteristic become more pronounced as n increases

The square of the Chebyshev polynomials have this property

$$H_{CC}(\omega)=\frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

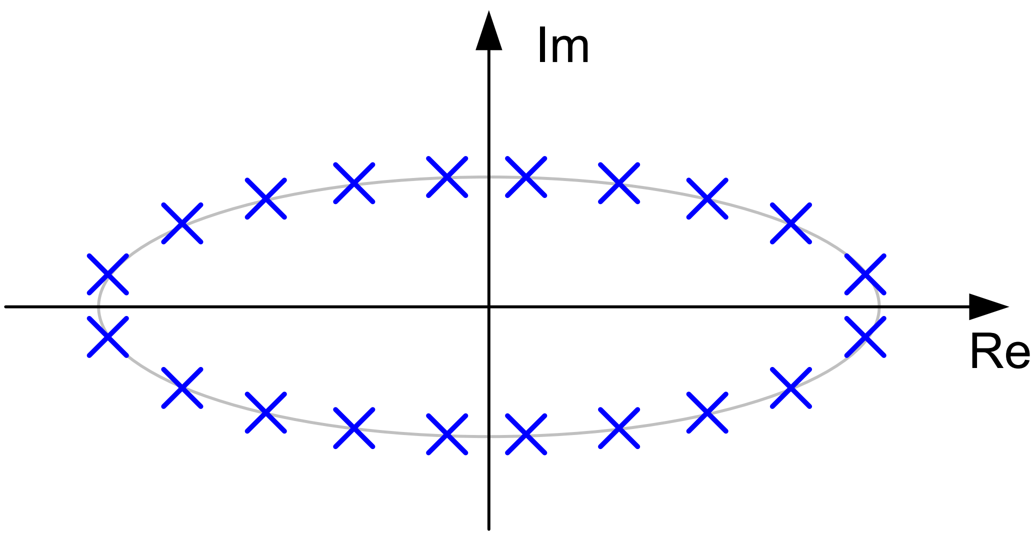
This is the magnitude squared approximating function of the Type 1 CC approximation (Often simply referred to as the Chebyshev approximation)

Chebyshev Approximations

Type 1

$$H_{CC}(\omega) = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

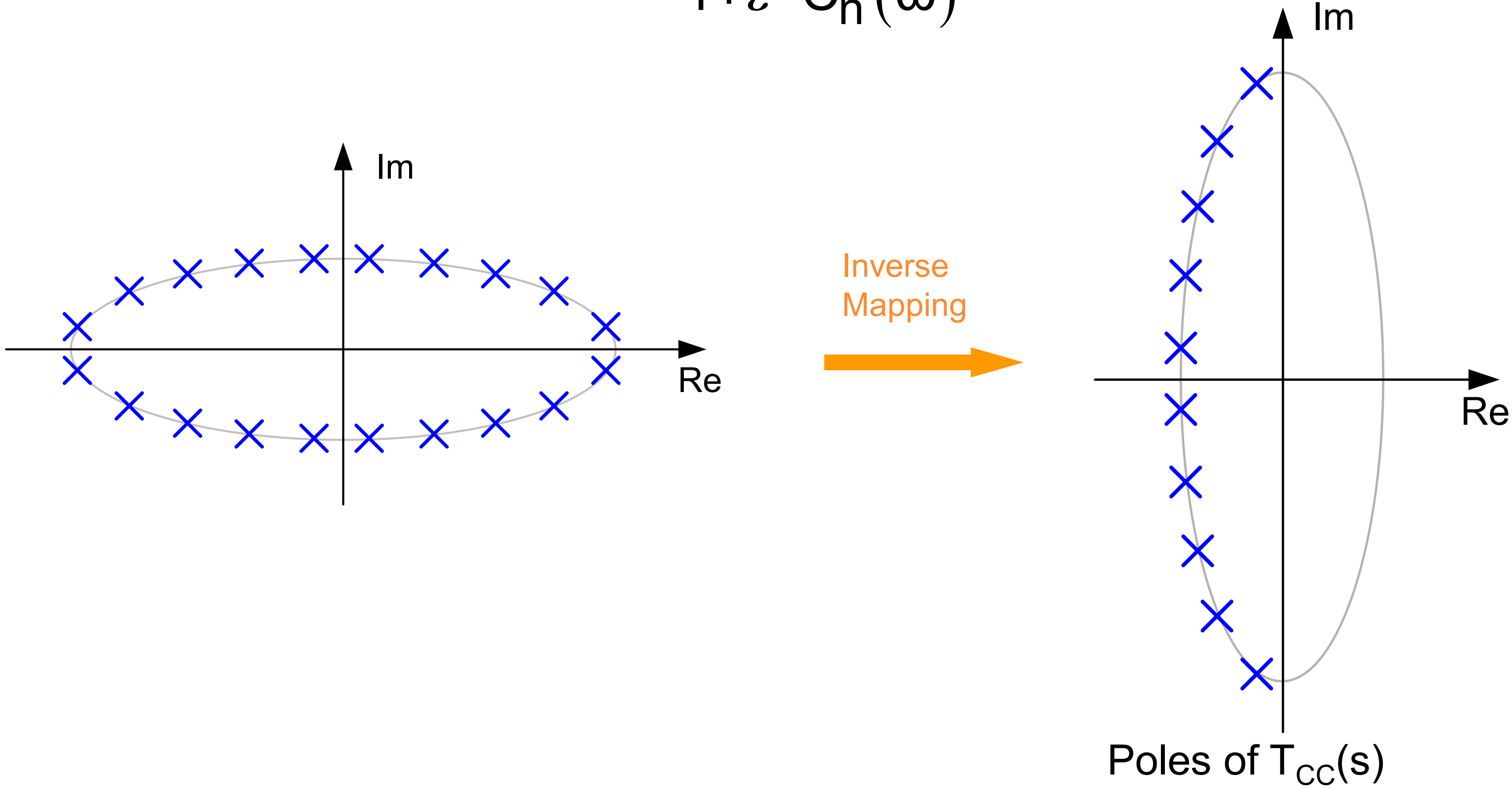
Poles of $H_{CC}(\omega)$ lie on an ellipse with none on the real axis



Chebyshev Approximations

Type 1

$$H_{CC}(\omega) = \frac{1}{1+\varepsilon^2 C_n^2(\omega)}$$

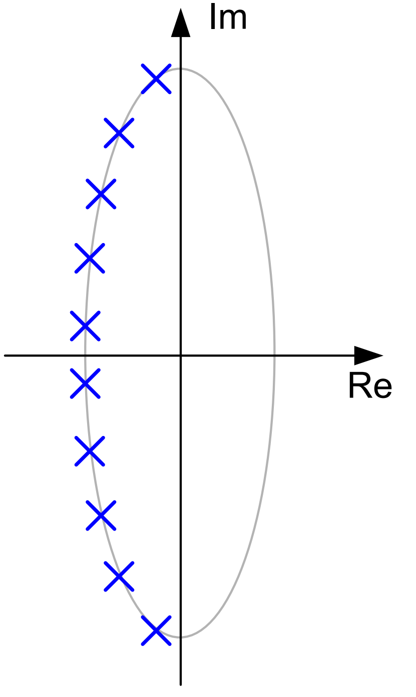


Chebyshev Approximations

Type 1

Equation for the ellipse:

$$\left[\frac{\alpha}{\sinh\left[\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right]}\right]^2 + \left[\frac{\beta}{\cosh\left[\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right]}\right]^2 = 1$$



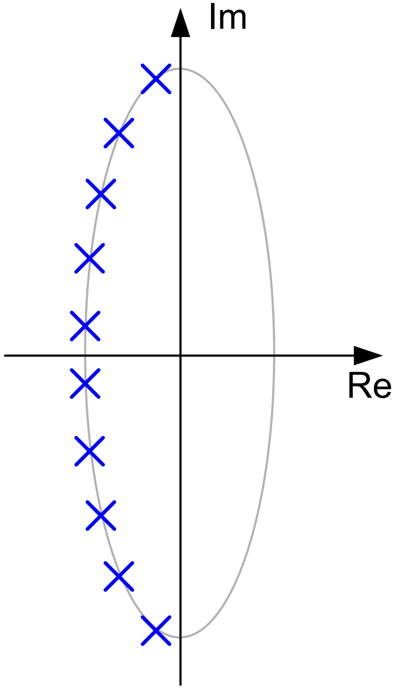
α is the real part and β is the imaginary part of points on locus

Ellipse Intersect Points for select n and ε

n	ε		Y int	X int
2	1		1.099	0.455
2	0.25		1.600	1.250
2	0.1		2.351	2.127
2	0.05		3.242	3.084
4	1		1.024	0.222
4	0.25		1.140	0.548
4	0.1		1.294	0.822
4	0.05		1.456	1.059
6	1		1.011	0.147
6	0.25		1.062	0.356
6	0.1		1.127	0.521
6	0.05		1.195	0.654
8	1		1.006	0.110
8	0.25		1.034	0.265
8	0.1		1.071	0.384
8	0.05		1.108	0.478

Chebyshev Approximations

Type 1



Poles of $T_{CC}(s)$

$$p_k = -\sin\left[\frac{\pi}{2n}(1+2k)\right]\sinh\left[\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right] \pm j\cos\left[\frac{\pi}{2n}(1+2k)\right]\cosh\left[\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right] \quad k=0....n-1$$

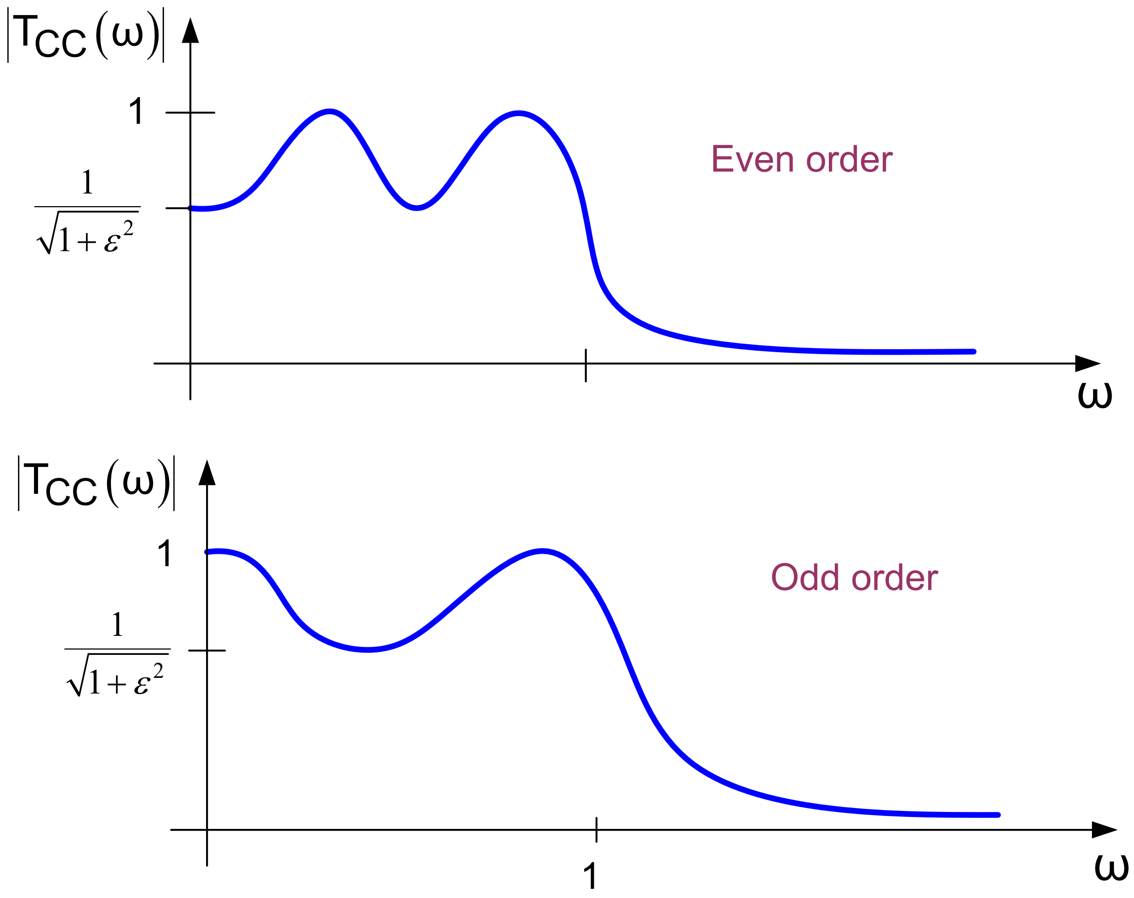
Properties of the ellipse

$$p_k = -\alpha_k \pm j\beta_k$$

$$\left[\frac{\alpha_k}{\sinh\left[\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right]}\right]^2 + \left[\frac{\beta_k}{\cosh\left[\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right]}\right]^2 = 1$$

Chebyshev Approximations

Type 1



- $|T_{CC}(0)|$ alternates between 1 and $\sqrt{\frac{1}{1+\varepsilon^2}}$ with index number
- Substantial pass band ripple
- Sharp transitions from pass band to stop band

Chebyshev Approximations

Type 1

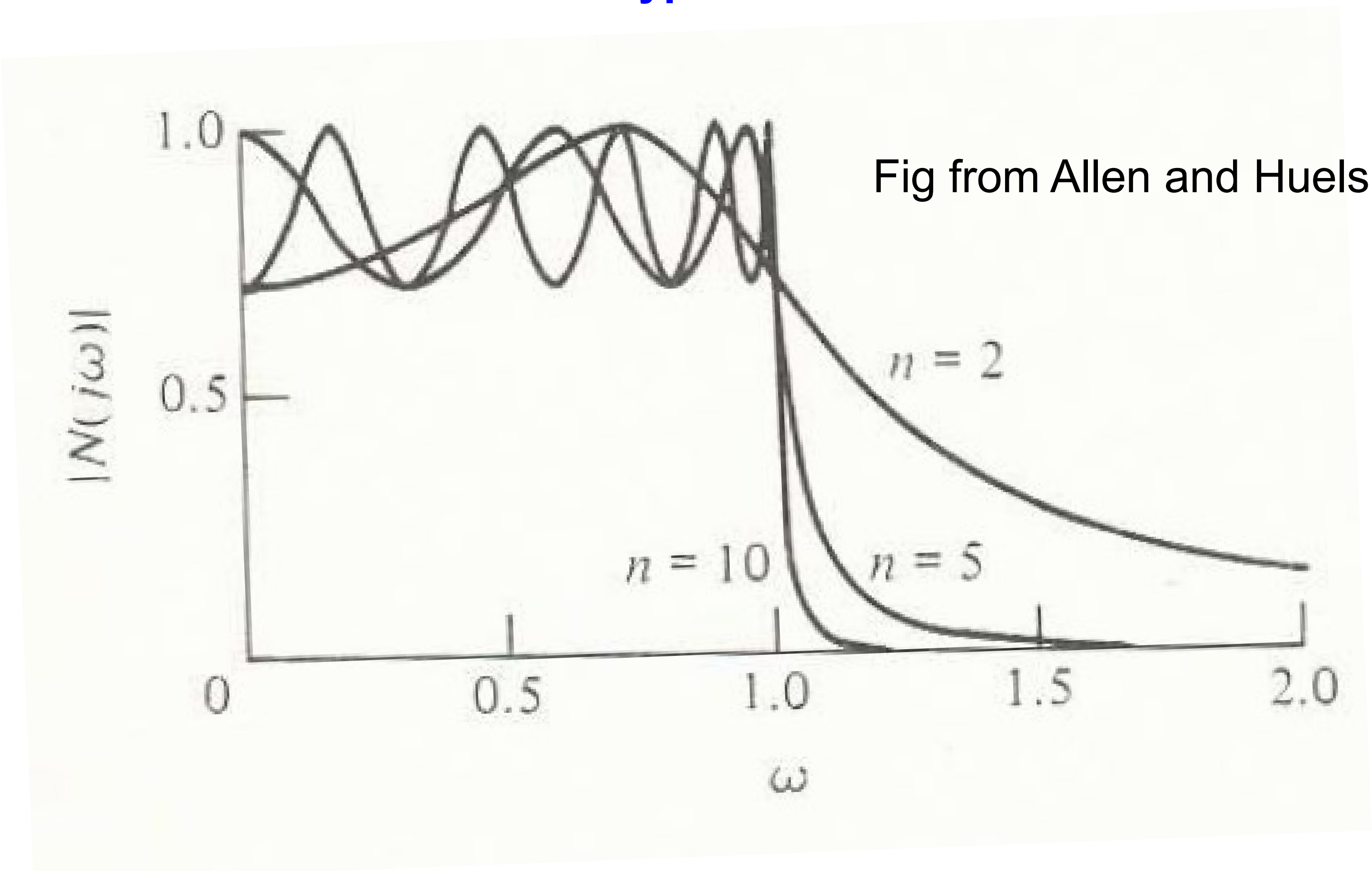


Fig from Allen and Huelsman

Sharp transitions from pass band to stop band

Chebyshev Approximations

Type 1

From Budak Text

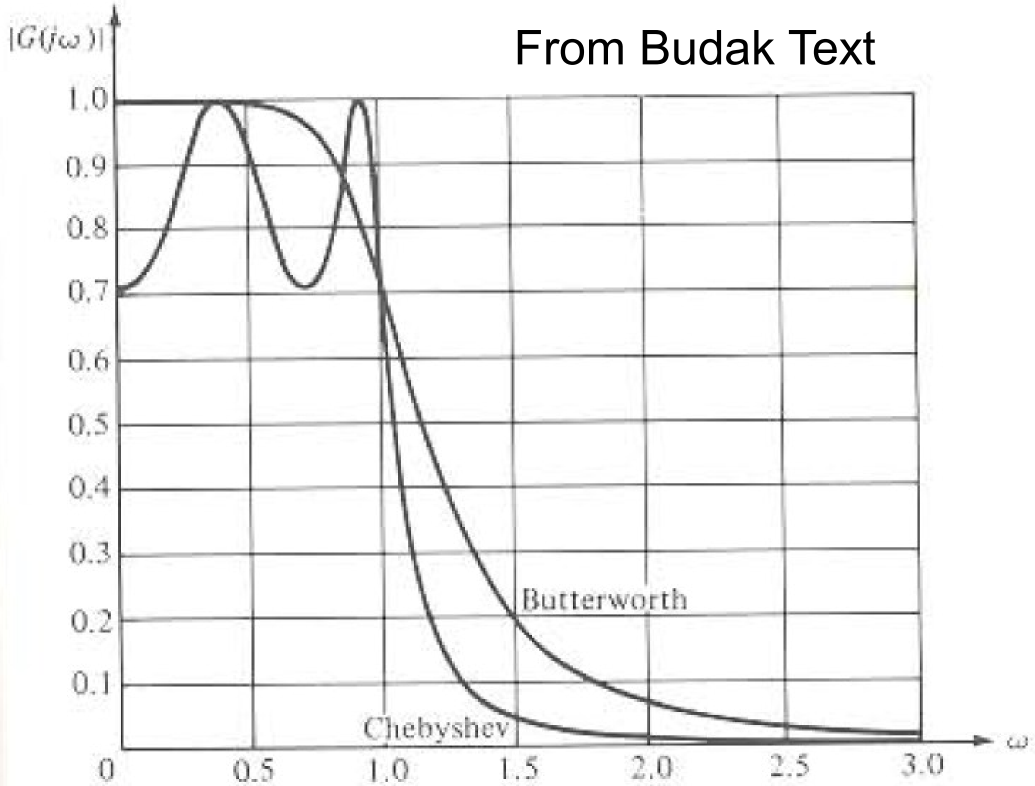


Fig. 17-6a Fourth-order Chebyshev and Butterworth magnitude characteristics

CC transition is much steeper than BW transition

Comparison of BW and CC Responses

- CC slope at band edge much steeper than that of BW

$$Slope_{CC}(\omega = 1) = -n^2 \frac{\varepsilon^2}{\left(1+\varepsilon^2\right)^{3/2}} = n \bullet Slope_{BW}(\omega = 1)$$

- Corresponding pole Q of CC much higher than that of BW
- Lower-order CC filter can often meet same band-edge transition as a given BW filter
- Both are widely used
- Cost of implementation of BW and CC for same order is about the same

Chebyshev Approximations

Type 1

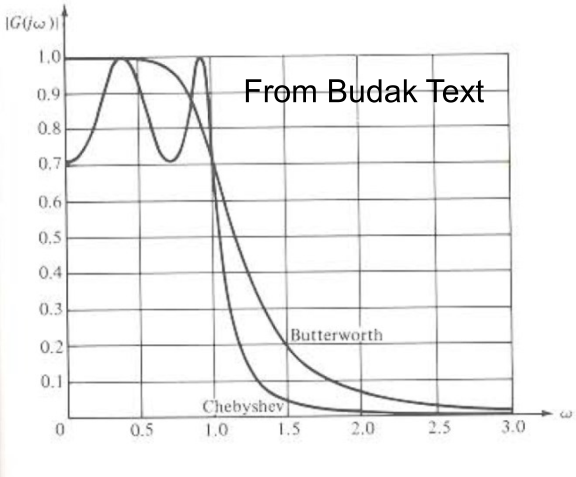


Fig. 17-6a Fourth-order Chebyshev and Butterworth magnitude characteristics

Analytically, it can be shown that, at the band-edge

$$\frac{d\left|T_{BW}\left(j\omega\right)\right|}{d\omega} = -n\frac{\varepsilon^2}{\left(1+\varepsilon^2\right)^{3/2}}$$

$$\frac{d\left|T_{CC}\left(j\omega\right)\right|}{d\omega} = -n^2\frac{\varepsilon^2}{\left(1+\varepsilon^2\right)^{3/2}}$$

CC slope is n times steeper than that of the BW slope

Chebyshev Approximations

Type 1

From Budak Text

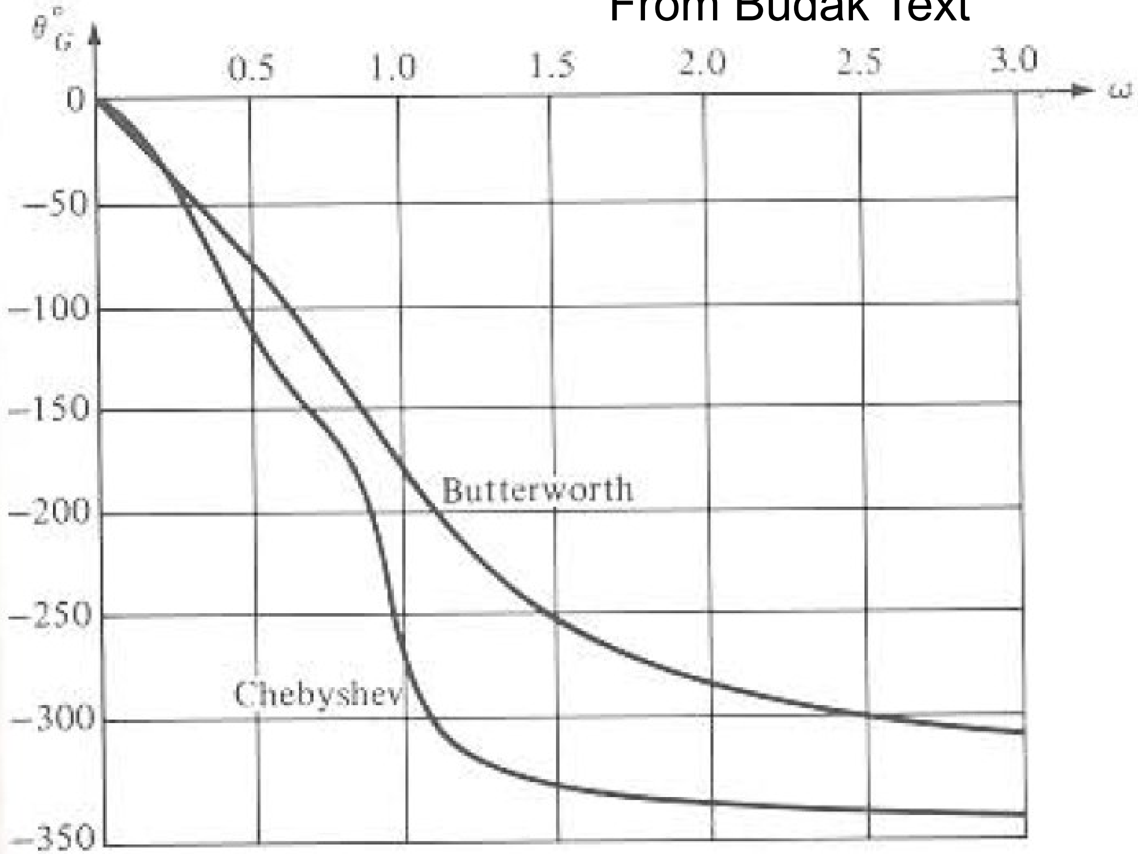
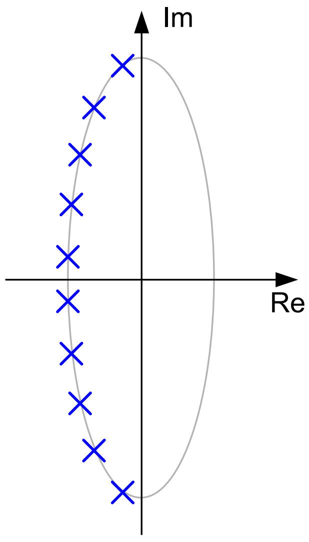


Fig. 17-6b Fourth-order Chebyshev and Butterworth phase characteristics

CC phase is much more nonlinear than BW phase

Chebyshev Approximations

Type 1



$$p_k = -\sin\left[\frac{\pi}{2n}(1+2k)\right]\sinh\left[\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right] \pm j\cos\left[\frac{\pi}{2n}(1+2k)\right]\cosh\left[\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right]$$

Maximum pole Q of CC approximation can be obtained by considering pole with index k=0

$$p_0 = -\sin\left[\frac{\pi}{2n}\right]\sinh\left[\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right] \pm j\cos\left[\frac{\pi}{2n}\right]\cosh\left[\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right]$$

$$p_0 = \alpha + j\beta$$

Recall

$$Q_{MAX} = \frac{\sqrt{\alpha^2+\beta^2}}{-2\alpha}$$

$$Q_{MAX,CC} = \left(\frac{1}{2\sin\left(\frac{\pi}{2n}\right)}\right)\sqrt{1+\left[\frac{\cos\left(\frac{\pi}{2n}\right)}{\sinh\left(\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right)}\right]^2}$$

Chebyshev Approximations

Type 1

Comparison of maximum pole Q of CC approximation with that of BW approximation

$$Q_{MAX,BW} = \frac{1}{2\sin\left(\frac{\pi}{2n}\right)} \qquad Q_{MAX,CC} = \left(\frac{1}{2\sin\left(\frac{\pi}{2n}\right)}\right)\sqrt{1+\left[\frac{\cos\left(\frac{\pi}{2n}\right)}{\sinh\left(\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right)}\right]^2}$$

$$Q_{MAX,CC} = Q_{MAX,BW}\sqrt{1+\left[\frac{\cos\left(\frac{\pi}{2n}\right)}{\sinh\left(\frac{1}{n}arc\sinh\left(\frac{1}{\varepsilon}\right)\right)}\right]^2}$$

Example – compare the Q's for $n=10$ and $\varepsilon=1$

$Q_{BW}=3.19$ $\qquad$ $Q_{CC}=35.9$

For large n, the CC filters have a very high pole Q !

Chebyshev Approximations

Type 2

$$H_{BW}(\omega)=\frac{1}{1+\varepsilon^2\omega^{2n}}$$

Butterworth

$$H(\omega)=\frac{1}{1+\varepsilon^2 F_n\left(\omega^2\right)}$$

A General Form

Another General Form

$$H(\omega)=\frac{1}{1+\dfrac{1}{\varepsilon^2 F_n\left(1/\omega^2\right)}}$$

- $F_n(\omega^{-2})$ bounded by 1 in the stop pass band and gets very large in pass-band
- These characteristic become more pronounced as n increases

$$H_{CC2}(\omega)=\frac{1}{1+\dfrac{1}{\varepsilon^2 C_n^2\left(1/\omega\right)}}$$

Note: The second general form is not limited to use of the Chebyshev polynomials

Chebyshev Approximations

Type 2

$$H_{CC2}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2\left(1/\omega\right)}}$$

- Equal-ripple in stop band
- Monotone in pass band
- Both poles and zeros present
- Poles of Type II CC are reciprocal of poles of Type I
- Zeros of Type II are inverse of the zeros of the CC Polynomials

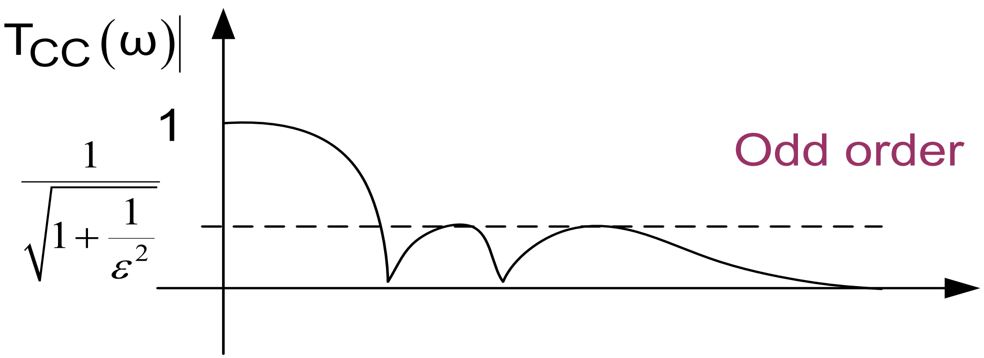
$$p_k = \frac{-1}{\sin\left[\frac{\pi}{2n}(1+2k)\right]\sinh\left[\frac{1}{n} arc\sinh\left(\frac{1}{\varepsilon}\right)\right] \pm j\cos\left[\frac{\pi}{2n}(1+2k)\right]\cosh\left[\frac{1}{n} arc\sinh\left(\frac{1}{\varepsilon}\right)\right]}$$

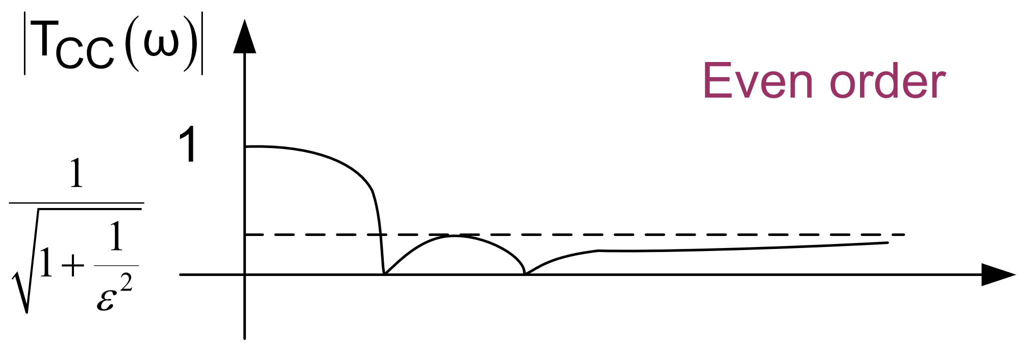
$$z_k = j\frac{1}{\cos\left(\frac{\pi}{2}\frac{(2k-1)}{n}\right)}$$

Chebyshev Approximations

Type 2

$$H_{CC2}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2\left(1/\omega\right)}}$$

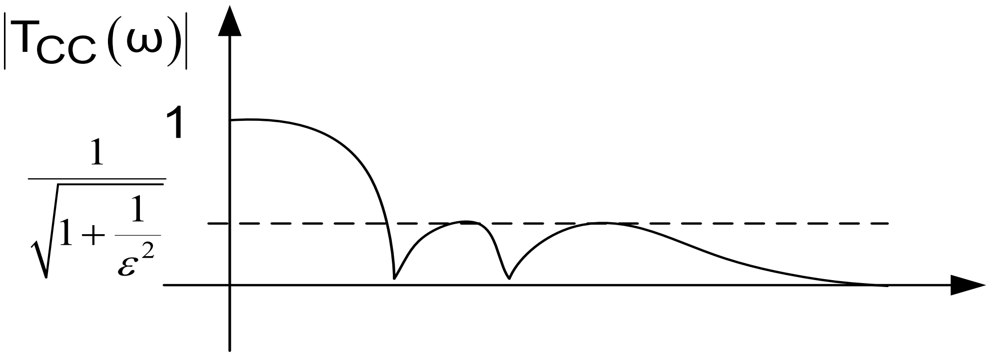




Chebyshev Approximations

Type 2

$$H_{CC2}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2\left(1/\omega\right)}}$$

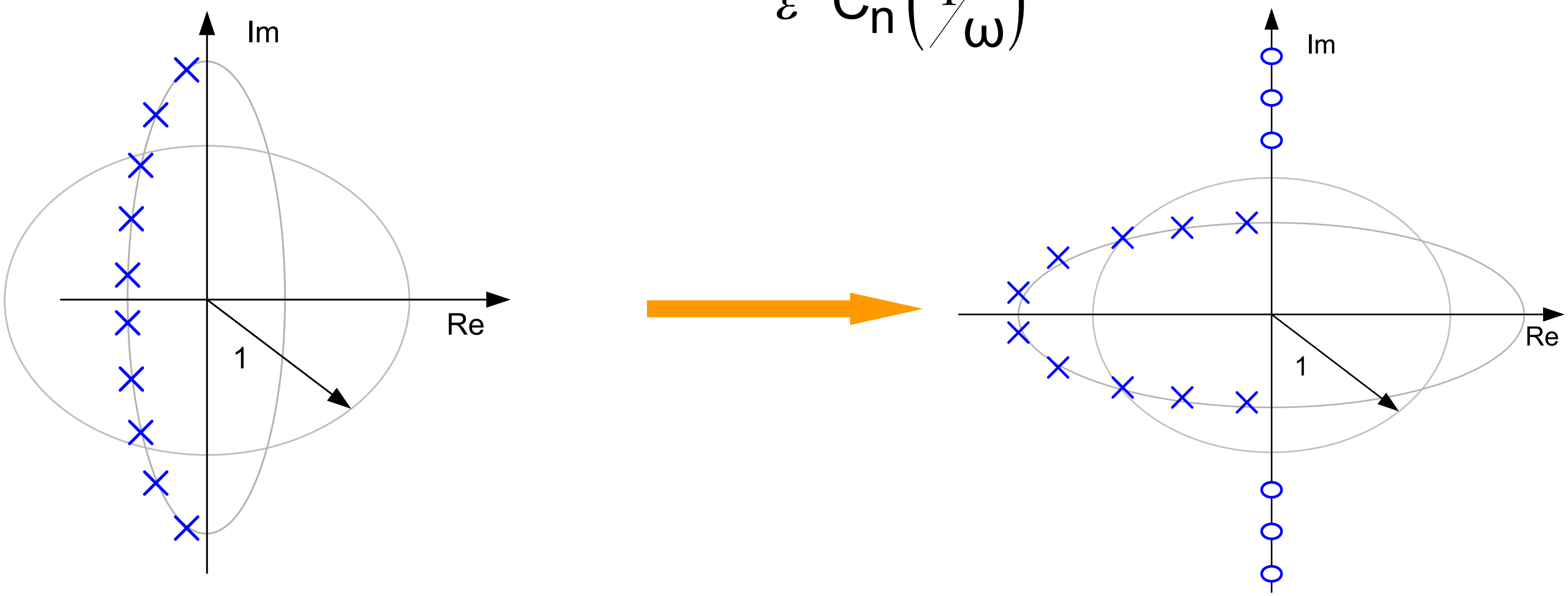


- Transition region not as steep as for Type 1
- Considerably less popular

Chebyshev Approximations

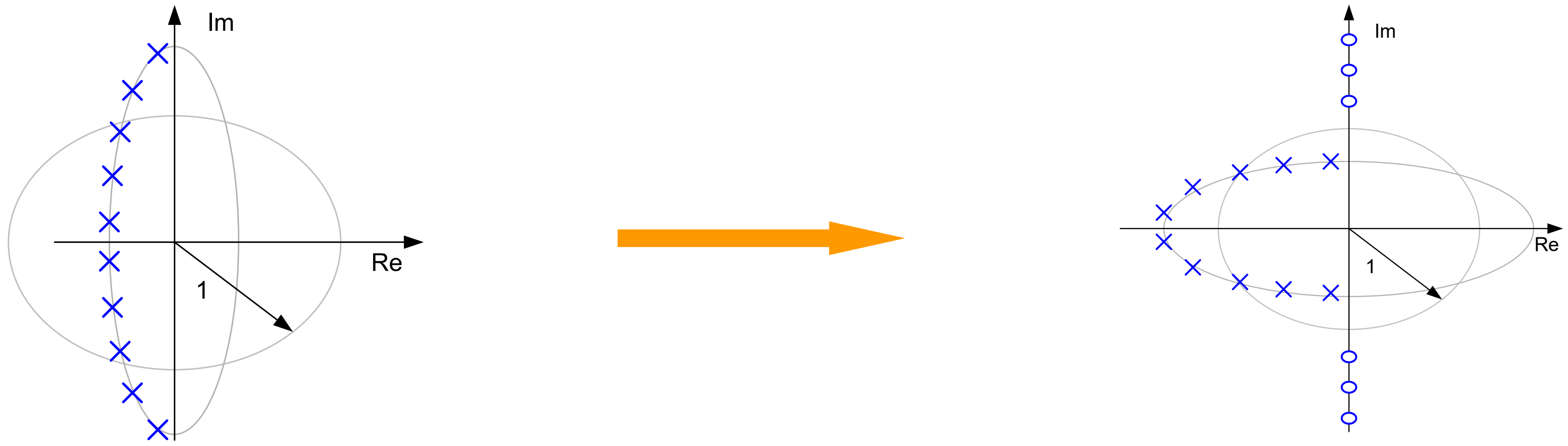
Type 2

$$H_{CC2}(\omega) = \frac{1}{1 + \frac{1}{\varepsilon^2 C_n^2\left(\frac{1}{\omega}\right)}}$$



- Pole Q expressions identical (within constant scale factor) since poles are reciprocals
- Maximum pole Q is just as high as for Type 1

Chebyshev Approximations



1821-1894

Was Chebyshev ahead of his time?

What role did Chebyshev have in developing the Chebyshev filter?

What role did Chebyshev have in developing the Chebyshev filter approximation?

Were we building filters when Chebshev did his work?

http://www.quadrivium.nl/history/history.htm
l

History of Filter Theory

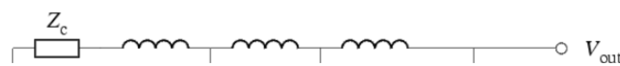
Michael I. Pupin

Around the year 1890 several people worked with the idea to improve the properties of long-distance transmission lines by inserting coils at regular intervals in these lines. Among those people were Vaschy and Heaviside. The results were discouraging at that time, and no real progress was made, until in 1899 M.I. Pupin investigated these cables [1]. He found that a line which contains coils at regular intervals can be represented by an equivalent uniform cable if the coils are spaced closely enough. The equivalence decreases if the distance between two adjacent coils is increased, and disappears altogether if this distance is larger than half the wave length of the signal that is propagated in the cable. By his thorough mathematical and experimental research, Pupin found that the damping in cables for telegraphy and telephony can be substantially reduced by judiciously inserting these coils, which has resulted in a widespread use of these so-called `Pupin lines' throughout the world.

The properties of these lines were further investigated by George A. Campbell. In 1903 he published some findings [2], among which a peculiar frequency-dependent effect of Pupin lines, namely that they have a well-defined critical frequency that marks a sudden change in the damping characteristics. Below this frequency the damping is low, and dependent only on the parasitic cable losses. If these losses are zero, the damping below the critical frequency is also zero. Above the critical frequency the damping is high, and almost independent of the cable losses. The transition at the critical frequency can be very sharp. The critical frequency itself is determined by the spacing of the coils and corresponds to a wave length equal to twice the distance between them.

This effect was used to answer the question of how many coils are to be inserted in a given length of cable, but it was also immediately clear that this effect could be utilized, and Campbell pointed out that he used this effect to eliminate harmonics in signal generators. In fact he used the cable as a lowpass filter, and he even mentioned the possibility of using the cable as a bandpass filter by replacing the coils by combinations of coils and capacitors.

A reel of cable is very large and therefore somewhat unwieldy as a filter, but the next step was so logical that it was undertaken independently in the same year (1915) in Germany by Karl Willy Wagner [3], and in America by Campbell [4]. The line was simulated by a ladder construction of impedances, an instance of which is shown in Figure 1.



Recall: Samuel Morse credited with introducing the concept of a telegraph in 1838

For almost a century the telegraph was the primary means for long-distance communications

Performance degraded with distance and it was observed that judicious placement of reactive elements along the cable could improve performance

Credited with inventing the concept of a filter in 1915

Michael I. Pupin

- Working on improving cables used for telegraph
- Nearly 75 years after the telegraph was introduced !!

Introduced Electrical Filters in 1915 to 1920 timeframe

Karl Willy Wagner

Publication in 1919

George A. Campbell

Patent in 1915

Transitional BW-Chebyshev Approximations

$$H(\omega)=\frac{1}{1+\varepsilon^2 F_n\left(\omega^2\right)}$$

General Form

Define $F_{BWk}=\omega^{2k}$ $F_{CCk}=C_n^2(\omega)$

Consider:

$$H(\omega)=\frac{1}{1+\varepsilon^2 F_{BWk}F_{CC(n-k)}} \qquad 0 \le k \le n$$

$$H(\omega)=\frac{1}{1+\varepsilon^2\left[(\theta)F_{BWk}+(1-\theta)F_{CC(n-k)}\right]} \qquad 0 \le \theta \le 1$$

- Other transitional approximations are possible
- Transitional approximations have some of the properties of both “parents”

Transitional BW-CC filters

$$H_{ABW}\left(\omega^2\right)=\frac{1}{1+\varepsilon^2\omega^{2n}}$$

$$H_{ACC}\left(\omega^2\right)=\frac{1}{1+\varepsilon^2\left(C_n\left(\omega\right)\right)^2}$$

$$H_{ATRAN1}\left(\omega^2\right)=\frac{1}{1+\varepsilon^2\left(\omega^{2k}\right)C_{n-k}^2\left(\omega\right)}$$

$$0\le k\le n$$

$$H_{ATRAN2}\left(\omega^2\right)=\frac{1}{1+\varepsilon^2\left[\theta\omega^{2n}+\left(1-\theta\right)C_n^2\left(\omega\right)\right]}$$

$$0\le \theta\le 1$$

Other transitional BW-CC approximations exist as well

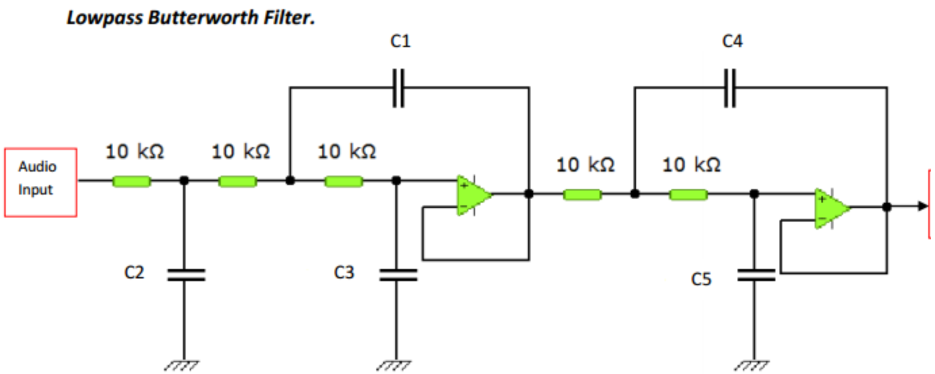
Transitional BW-CC filters

$$H_{ATRAN\,1}\left(\omega^2\right)=\frac{1}{1+\varepsilon^2\left(\omega^{2k}\right)C_{n-k}^2\left(\omega\right)}$$

$$H_{ATRAN\,2}\left(\omega^2\right)=\frac{1}{1+\varepsilon^2\left[\theta\omega^{2n}+\left(1-\theta\right)C_n^2\left(\omega\right)\right]}$$

Transitional filters will exhibit flatness at ω=0, passband ripple, and intermediate slope characteristics at band-edge

Distinguish Between Circuit and Approximation

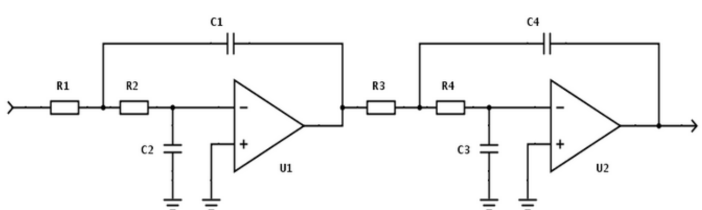


http://www.egr.msu.edu/classes/ece480/capstone/fall11/group02/web/Documents/How%20to%20Design%2010%20kHz%20filter-Vadim.pdf

Active Butterworth Lowpass Filter Calculator

Unity Gain in the Passband, 24 dB / Octave, 2 x 2nd order

- Maximally flat near the center of the band
- Smooth transition from Passband to Stopband
- Moderate out of band Rejection
- Low Group Delay variation near center of band



http://www.changpuak.ch/electronics/Butterworth_Lowpass_active_24dB.php

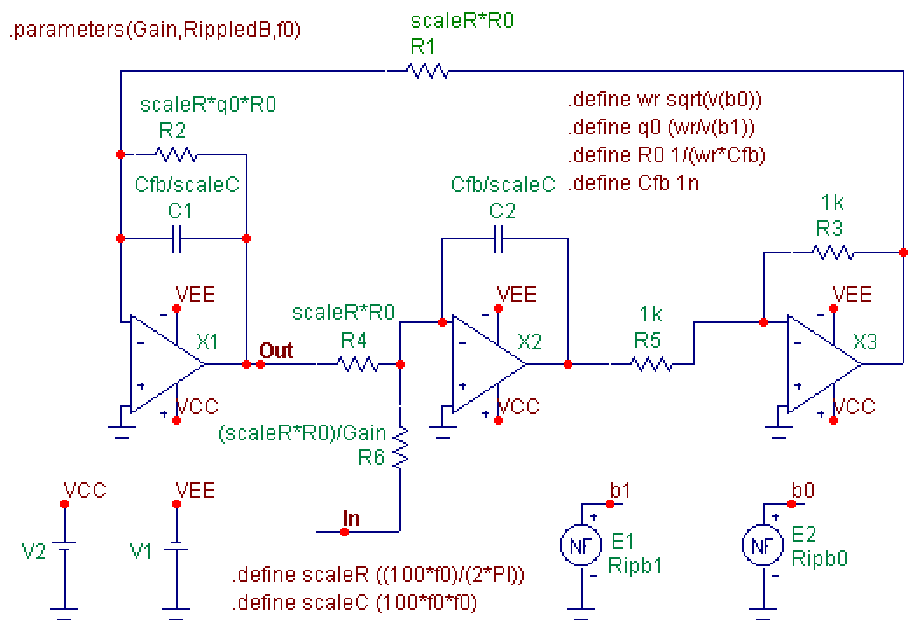
Note that what distinguishes between different filter approximations having the same number of cc poles and zeros and the same number of real axis poles and zeros is the component values of a given circuit, not the filter architecture itself

Chebyshev Approximations

from Spectrum Software:

Chebyshev Filter Macro

Filters are a circuit element that seem to mesh perfectly with the macro capability of Micro-Cap. The macro capability is designed to produce components that can be varied through the use of parameters. Most filters consist of a basic structure whose component values can be modified through the use of well known equations. A macro component can be created that represents a specific filter's type, order, response, and implementation. The circuit below is the macro circuit for a low pass, 2nd order, Chebyshev filter with Tow-Thomas implementation.



- Note that this is introduced as a Chebyshev filter, the source correctly points out that it implements the CC filter in a specific filter topology
- It is important to not confuse the approximation from the architecture and this Tow-Thomas Structure can be used to implement either BW or CC functions only differing in the choice of the component values

Stay Safe and Stay Healthy !

End of Lecture 10